The natural definition of the quantum dynamical entropy in the framework of deformation quantization

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It is shown how, in the framework of deformation quantization, the quantum dynamical entropy may be simply defined as the Kolmogorov-Sinai entropy of the quantum flow.

Contents

I. ACKNOWLEDGMENTS

I would like to thank Gianni Jona-Lasinio and Vittorio de Alfaro for many precious teachings. Of course nobody among them has any responsibility as to any eventual mistake contained in this paper.

II. DEFORMATION QUANTIZATION

The core of deformation quantization [1], [2], [3], [4], [5], [6] consists in the idea that quantization may be understood as a deformation of the structure of the algebra of classical observables, rather than a radical change in the nature of the observables.

To formalize the mathematical essence of its structure let us start from the following [7], [8]:

Definition II.1

symplectic manifold:
   a couple (M, ω) where:

- M is a differentiable manifold

- ω is a symplectic (i.e. closed, non-degenerate) 2-form over M

Given a symplectic manifold (M, ω) and a map $f \in C^\infty(M)$:

Definition II.2

hamiltonian vector field generated by f:

$$X_f \in \Gamma(TM) : i_{X_f}\omega = df \tag{2.1}$$

Given two maps $f, g \in C^\infty(M)$:

Definition II.3

Poisson bracket of f and g:

$$\{f, g\} = \omega(X_f, X_g) \tag{2.2}$$

Let us recall that $(C^\infty(M), \{\cdot, \cdot\})$ is a Lie algebra.
   Let us then define the following:

Definition II.4

$\star$-product of f and g:

$$f \star g := \exp[\frac{i\hbar}{2}\{f, g\}] \tag{2.3}$$

The $\star$-product may be used to introduce the following:

Definition II.5

Moyal bracket of f and g:

$$\{f, g\}_\hbar := \frac{f \star g - g \star f}{i\hbar} \tag{2.4}$$

$(C^\infty(M), \{\cdot, \cdot\}_\hbar)$ is called the *deformation quantization* of the symplectic manifold (M, ω).
   Let us then define the following:

Definition II.6

quantum flow [1] *generated by* $H \in C^\infty(M)$:
   the flow $\{U_t(H)\}_{t \in \mathbb{R}}$ over $C^\infty(M)$ ruled by the following *Moyal equation:*

$$\frac{\partial f}{\partial t} = \{H, f\}_\hbar \tag{2.5}$$

The classical limit consists in the fact that, for $\hbar \to 0$, the Moyal bracket reduces to the Poisson bracket and hence the Moyal equation reduces to the *Liouville equation:*

$$\frac{\partial f}{\partial t} = \{H, f\} \tag{2.6}$$

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[1] Except from the cases in which the Schrödinger picture will be explicitly indicated we will work in the Heisenberg picture in which observables evolve with time.

III.   KOLMOGOROV-SINAI ENTROPY OF A FLOW

Let (X, σ, μ) be a classical probability space and let us introduce the following:

Definition III.1

finite partitions of (X, σ, μ):

$$\mathcal{P}(X, \sigma, \mu) := \{P = \{A_i\}_{i=1}^{n(P)} : n(P) \in \mathbb{N} \,, \; A_i \in \sigma \; i = 1, \cdots, n(P)$$
$$A_i \cap A_j = \emptyset \; i, j = 1, \cdots, n(P) : i \neq j \; \mu(X - \cup_{i=1}^{n(P)} A_i) = 0\} \quad (3.1)$$

Remark III.1

Beside its abstract, mathematical formalization, the definition III.1 has a precise operational meaning.

Given the classical probability space (X, σ, μ) let us suppose to make an experiment on the probabilistic universe it describes using an instrument whose resolutive power is limited in that it is not able to distinguish events belonging to the same atom of a partition $P = \{A_i\}_{i=1}^n \in \mathcal{P}(X, \mu)$.

Consequentially the outcome of such an experiment will be a number

$$r \in \{1, \cdots, n\} \quad (3.2)$$

specifying the observed atom A_r in our coarse-grained observation of (X, σ, μ).

We will call such an experiment an *operational observation of* (X, σ, μ) *through the finite partition* P or, more concisely, a *P-experiment*.

The probabilistic structure of the operational observation of (X, σ, μ) through a finite partition $P \in \mathcal{P}(X, \sigma, \mu)$ is enclosed in the following:

Definition III.2

probability measure of the P-experiment:

$$\mu_P := \mu|_{\sigma(P)}$$

where $\sigma(P) \subset \sigma$ is the σ-algebra generated by P.

Definition III.3

coarsest refinement of $A = \{A_i\}_{i=1}^n$ *and* $B = \{B_j\}_{j=1}^m \in \mathcal{P}(X, \sigma, \mu)$:

$$A \vee B \in \mathcal{P}(X, \sigma, \mu)$$
$$A \vee B \equiv \{A_i \cap B_j \; i = 1, \cdots, n \; j = 1, \cdots, m\} \quad (3.3)$$

Clearly $\mathcal{P}(X, \sigma, \mu)$ is closed under coarsest refinements.

Let us now introduce the following:

Definition III.4

entropy of $P = \{A_i\}_{i=1}^{n(p)} \in \mathcal{P}(X, \sigma, \mu)$:

$$H(P) := -\sum_{i=1}^{n(P)} \mu_P(A_i) \log_2 \mu_P(A_i) \quad (3.4)$$

The more abstract definition of a classical dynamical system is the following:

Definition III.5

classical dynamical system :
  a couple $((X, \sigma, \mu), T)$ such that:

- (X, σ, μ) is a classical probability space

- $T : X \mapsto X$ is such that:

$$\mu \circ T^{-1} \; = \; \mu \tag{3.5}$$

Given a classical dynamical system $CDS := ((X, \sigma, \mu), T)$, the T^{-1}-invariance of μ implies that the finite partitions $P = \{A_i\}_{i=1}^{n} \in \mathcal{P}(X, \sigma, \mu)$ and $T^{-1}P$ have equal probabilistic structure. Consequentially the P *experiment* and the $T^{-1}P$-*experiment* are replicas, not necessarily independent, of the same experiment made at successive times.
  In the same way the $\vee_{k=0}^{n-1} T^{-k}P$-*experiment* is the compound experiment consisting in n replications $P, T^{-1}P, , \cdots, T^{-(n-1)}P$ of the experiment corresponding to $A \in \mathcal{P}(X, \sigma, \mu)$.
  The rate of classical information for replication we obtain in this compound experiment is clearly:

$$\frac{1}{n} H(\vee_{k=0}^{n-1} T^{-k}P)$$

It may be proved (cfr. e.g. the second paragraph of the third chapter of [9]) that when n grows this rate of classical information acquired for replication converges, so that the following quantity:

$$h(P, T) \; := \; lim_{n \to \infty} \frac{1}{n} H(\vee_{k=0}^{n-1} T^{-k}P) \tag{3.6}$$

does exist.
  Clearly $h(P, T)$ gives the asymptotic rate of production of classical information for replication of the P-experiment.

Definition III.6

Kolmogorov-Sinai entropy of CDS:

$$h(CDS) \; := \; sup_{P \in \mathcal{P}(X, \sigma, \mu)} h(P, T) \tag{3.7}$$

By definition we have clearly that:

$$h(CDS) \; \geq \; 0 \tag{3.8}$$

Definition III.7

CDS is chaotic:

$$h(CDS) \; > \; 0 \tag{3.9}$$

Let us now introduce the straightforward generalization of these notions to flows:

Definition III.8

classical flow:
  a couple $((X, \sigma, \mu), \{T_t\}_{t \in \mathbb{R}})$ such that:

- $((X, \sigma, \mu), T_t)$ is a classical dynamical system for every $t \in \mathbb{R}$

- $\{T_t\}_{t \in \mathbb{R}}$ is an abelian group

Given a classical flow $F = ((X, \sigma, \mu), \{T_t\}_{t \in \mathbb{R}})$ it is natural to define its dynamical entropy in the following way :

Definition III.9

Kolmogorov-Sinai entropy of F:

$$h(F) \; := \; h[((X, \sigma, \mu), T_1)] \tag{3.10}$$

IV. QUANTUM DYNAMICAL ENTROPY

The problem of characterizing correctly the notion of quantum dynamical entropy, i.e. the quantum analogue of the Kolmogorov-Sinai entropy, has led to a plethora of candidate notions, the more famous ones being the *Connes-Narnhofer-Thirring entropy* [10], [11] and the *Alicki-Fannes-Lindblad entropy* [12], [13].

Let us remark, anyway, that no one of these proposals has a clear information-theoretic meaning comparable to that of the Kolmogorov-Sinai entropy.

From the viewpoint of deformation quantization, from the other side, it appears natural to define the quantum dynamical entropy simply as the Kolmogorov-Sinai entropy of the quantum flow $U_t(H)$.

In order to implement technically such an idea we have to introduce some notion concerning algebraic dynamical systems.

Let us recall first of all the following:

Definition IV.1

algebraic probability space:
    a couple (A, ω) such that:

- A is a $W^\star$-algebra

- $\omega \in S(A)$ is a state over A

The algebraic probability space (A, ω) is said to be commutative whether A is commutative.

Definition IV.2

algebraic dynamical system:
    a triple (A, ω, τ) such that:

- (A, ω) is an algebraic probability space

- τ is an endomorphism of A ω-preserving (i.e. such that $\omega \circ \tau = \omega$).

The algebraic dynamical system (A, ω, τ) is said to be commutative whether (A, ω) is commutative.

A classical dynamical system $((X, \sigma, \mu), T)$ may be seen as the commutative algebraic dynamical system $(L^\infty(X, \mu), \omega_\mu, \Theta_\tau)$ where:

Definition IV.3

state over $L^\infty(M, \mu)$ associated to μ:

$$\omega_\mu(f) \; := \; \int_M f d\mu \tag{4.1}$$

Definition IV.4

endomorphism of $L^\infty(M, \mu)$ associated to τ:

$$\Theta_\tau(f) \; := \; f \circ \tau^{-1} \tag{4.2}$$

Given a commutative $W^\star$-algebra A let us denote by $\mathcal{F}(A)$ the collection of finite dimensional subalgebras of A.

Given a state $\omega \in S(A)$ and a subalgebra $N \in \mathcal{F}(A)$ having $\{n_i\}_{i=1}^k$ as minimal projections:

Definition IV.5

entropy of ω with respect to N:

$$H_\omega(N) \; := \; -\sum_{i=1}^k \omega(n_i) \log_2 \omega(n_i) \tag{4.3}$$

Given $N_1, N_2 \in \mathcal{F}(A)$:

Definition IV.6

coarsest refinement of N_1 and N_2:

> the subalgebra $N_1 \vee N_2 \in \mathcal{F}(A)$ having as miminal projections the product of the minimal projections of, respectively, N_1 and N_2

Given a commutative algebraic dynamical system $ADS := (A, \omega, \tau)$:

Definition IV.7

Kolmogorov-Sinai entropy of ADS:

$$h(ADS) := \sup_{N \in \mathcal{F}(A)} \lim_{n \to \infty} \frac{1}{n} H(N \vee \tau(N) \vee \cdots \vee \tau^{n-1}(N)) \tag{4.4}$$

Let us now introduce the straightforward generalization of these notions to commutative algebraic flows.

Definition IV.8

algebraic flow:
a triple $(A, \omega, \{\tau_t\}_{t \in \mathbb{R}})$ such that:

- (A, ω, τ_t) is an algebraic dynamical system for every $t \in \mathbb{R}$

- $\{\tau_t\}_{t \in \mathbb{R}}$ is an abelian group

Clearly an algebraic flow is said to be commutative whether the involved $W^\star$-algebra is commutative.
Given a commutative algebraic flow $AF = (A, \omega, \{\tau_t\}_{t \in \mathbb{R}})$:

Definition IV.9

Kolmogorov-Sinai entropy of AF:

$$h(AF) := h[(A, \omega, \tau_1)] \tag{4.5}$$

Remark IV.1

Let us remark that, given a classical flow $CF := ((X, \sigma, \mu), \{\tau_t\}_{t \in \mathbb{R}})$ and the associated commutative algebraic flow $AF := (L^\infty(X, \mu), \omega_\mu, \{\Theta_{\tau_t}\}_{t \in \mathbb{R}})$ one has that:

$$h(AF) = h(CF) \tag{4.6}$$

Since from the other side, owing to the Gelfand isomorphism, given a commutative algebraic flow $(A, \omega, \{\tau\}_{t \in \mathbb{R}})$ there exists a classical flow $((X, \sigma, \mu), \{T_t\}_{t \in \mathbb{R}})$ such that $A = L^\infty(X, \mu)$, $\omega = \omega_\mu$ and $\tau_t = \Theta_{T_t}$, the definition IV.9 is conceptually equivalent to the definition III.9 and has, in particular, the clear information-theoretic meaning explained in the previous section.

Let us now observe that the classical flow of an hamiltonian dynamical system having as phase space the symplectic manifold (M, ω) and as hamiltonian $H \in C^\infty(M)$ may be seen as the commutative algebraic flow $F(H) := (L^\infty(M, \mu_{Liouville}), \omega_{\mu_{Liouville}}, \lim_{\hbar \to 0} \{U_t(H)\}_{t \in \mathbb{R}})$, where $\mu_{Liouville}$ is the following:

Definition IV.10

Liouville measure on (M, ω):

$$\mu_{Liouville} := \wedge_{i=1}^{\frac{dim M}{2}} \omega \tag{4.7}$$

and where the fact that the Liouville state $\omega_{\mu_{Liouville}}$ is invariant under the hamiltonian flow $\lim_{\hbar \to 0} \{U_t(H)\}_{t \in \mathbb{R}}$ is owed to the fact that introduced the following:

Definition IV.11

group of the symplectic diffeomorphisms of (M, ω):

$$Symp(M, \omega) := \{\phi \in Diff(M) : \{f, g\} \circ \phi = \{f \circ \phi, g \circ \phi\} \ \forall f, g \in C^\infty(M)\} \tag{4.8}$$

the hamiltonian flow in the Schrödinger picture is made of symplectic diffeomorphisms and that the Liouville measure $\mu_{Liouville}$ is $Symp(M, \omega)$-invariant.

Using the Moyal bracket instead of the Poisson bracket it is then natural to introduce the following:

Definition IV.12

group of the quantum symplectic diffeomorphisms of (M, ω):

$$Symp_\hbar(M, \omega) := \{\phi \in Diff(M) : \{f, g\}_\hbar \circ \phi = \{f \circ \phi, g \circ \phi\}_\hbar \ \forall f, g \in C^\infty(M)\} \tag{4.9}$$

The definition II.3 naturally suggests to introduce the following:

Definition IV.13

deformed symplectic form on (M, ω):
the 2-form $\omega_\hbar$ on M such that:

$$\{f, g\}_\hbar = \omega_\hbar(X_f, X_g) \ \forall f, g \in C^\infty(M) \tag{4.10}$$

in terms of which it is natural to introduce the following:

Definition IV.14

Moyal measure:

$$\mu_{Moyal} := \wedge_{i=1}^{\frac{dim M}{2}} \omega_\hbar \tag{4.11}$$

Let us now observe that the quantum flow generated by the hamiltonian $H \in C^\infty(M)$ may be seen as the commutative algebraic flow $F_\hbar(H) := ((L^\infty(M, \mu_{Moyal}), \omega_{\mu_{Moyal}}, \{U_t(H)\}_{t \in \mathbb{R}})$ where the fact that the Moyal state $\omega_{\mu_{Moyal}}$ is invariant under the quantum flow $\{U_t(H)\}_{t \in \mathbb{R}}$ is owed to the fact that the quantum flow in the Schrödinger picture is made of quantum symplectic diffeomorphisms and that, by construction, the Moyal measure μ_{Moyal} is $Symp_\hbar(M, \omega)$-invariant.

We can finally define the quantum dynamical entropy as the Kolmogorov-Sinai entropy of the quantum flow:

Definition IV.15

quantum dynamical entropy of the quantum system with hamiltonian $H \in C^\infty(M)$:

$$h_\hbar(H) := h(F_\hbar(H)) \tag{4.12}$$

Denoted by $h(H) := h(F(H))$ the Kolmogorov-Sinai entropy of the classical system with hamiltonian $H \in C^\infty(M)$ we will say, according to the definition III.7, that:

Definition IV.16

the classical system with hamiltonian $H \in C^\infty(M)$ is chaotic:

$$h(H) > 0 \tag{4.13}$$

while we will say that:

Definition IV.17

the quantum system with hamiltonian $H \in C^\infty(M)$ is quantum-chaotic:

$$h_\hbar(H) > 0 \tag{4.14}$$

Remark IV.2

The definition IV.17 furnishes a new characterization of what Quantum Chaos is endowed with a clear information-theoretic meaning.

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